Roots and Weyl Groups

Yan

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1 Root Decomposition and Killing Form

For a complex semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, the root decomposition is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{X \mid [H, X] = \alpha(H)X, \forall H \in \mathfrak{h}\}.$$

The Killing form $B(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)$ is non-degenerate, inducing an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$:

$$\alpha(H) = B(H_\alpha, H), \quad \forall H \in \mathfrak{h}.$$

For $\alpha \in \Delta$, normalize H_α by $\alpha(H_\alpha) = 2$ and select $E_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ satisfying

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \quad B(E_\alpha, E_{-\alpha}) = 1,$$

forming an $\mathfrak{sl}(2)$-triple with commutation relations:

$$[H_\alpha, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}, \quad [E_\alpha, E_{-\alpha}] = H_\alpha.$$

2 Root System Structure Theorem

Theorem 2.1. *The set of roots $\Delta \subseteq \mathfrak{h}^*$ of a complex semisimple Lie algebra $\mathfrak{g}$ forms an abstract root system in $\mathfrak{h}^*$ equipped with the inner product induced by the Killing form.*

Proof. Since Δ is finite and spans $\mathfrak{h}^*$, verify root system axioms:

For $\alpha, \beta \in \Delta$, define reflection

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

Representation theory of $\mathfrak{sl}(2)$ ensures symmetry, thus $\sigma_\alpha(\beta) \in \Delta$.

For multiples, finite-dimensionality and eigenvector independence imply $c\alpha \in \Delta$ only if $c = \pm 1$. □

3 Abstract Root Systems and Weyl Groups

An abstract root system is a finite set $R \subset E$ satisfying:

(R1) R spans E;

(R2) $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$;

(R3) reflections $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ preserve R;

(R4) if $c\alpha \in R$, then $c = \pm 1$.

The Weyl group is defined as

$$W = \langle s_\alpha \mid \alpha \in R \rangle \subseteq O(E),$$

is finite, and stabilizes R.

4 Rank 2 Root Systems Examples

Type $A_1 \cup A_1$: Four roots $\pm\alpha, \pm\beta$ orthogonal and equal length. Weyl group:

$$W \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Type A_2: Six roots $\pm\alpha, \pm\beta, \pm(\alpha + \beta)$ forming a regular hexagon with equal length and angle 120°. Weyl group:

$$W \cong S_3.$$

Type B_2: Eight roots $\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(\alpha + 2\beta)$ with lengths differing by $\sqrt{2}$. Weyl group:

$$W \cong D_4 \text{ (dihedral group of order 8)}.$$

Type G_2: Twelve roots $\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)$ with lengths differing by $\sqrt{3}$. Weyl group:

$$W \cong D_6 \text{ (dihedral group of order 12)}.$$